

ASYMPTOTIC RESULTS FOR BIFURCATING RANDOM COEFFICIENT AUTOREGRESSIVE PROCESSES

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ABSTRACT. The purpose of this paper is to study the asymptotic behavior of the weighted least square estimators of the unknown parameters of random coefficient bifurcating autoregressive processes. Under suitable assumptions on the immigration and the inheritance, we establish the almost sure convergence of our estimators, as well as a quadratic strong law and central limit theorems. Our study mostly relies on limit theorems for vector-valued martingales.

1. INTRODUCTION

In this paper, we will study random coefficient autoregressive bifurcating autoregressive processes (RCBAR). Those processes are an adaptation of random coefficient autoregressive processes (RCAR) to binary tree structured data. We can also see those processes as the combination of RCAR processes and bifurcating autoregressive processes (BAR). RCAR processes have been first studied by Nicholls and Quinn [17, 18] while BAR processes have been first investigated by Cowan and Staudte [5]. Both inherited and environmental effects are taken into consideration in RCBAR processes in order to explain the evolution of the characteristic under study. The binary tree structure could lead us to take cell division as an example.

More precisely, the first-order RCBAR process is defined as follows. The initial cell is labelled 1 and the offspring of the cell labelled n are labelled $2n$ and $2n + 1$. Denote by X_n the characteristic of individual n . Then, the first-order RCBAR process is given, for all $n \geq 1$, by

$$\begin{cases} X_{2n} &= a_n X_n + \varepsilon_{2n} \\ X_{2n+1} &= b_n X_n + \varepsilon_{2n+1} \end{cases}$$

The environmental effect is given by the driven noise sequence $(\varepsilon_{2n}, \varepsilon_{2n+1})_{n \geq 1}$ while the inherited effect is given by the random coefficient sequence $(a_n, b_n)_{n \geq 1}$. The cell division example leads us to consider that ε_{2n} and ε_{2n+1} are correlated since the environmental effect on two sister cells can reasonably be seen as correlated.

2010 *Mathematics Subject Classification.* Primary 60F15; Secondary 60F05, 60G42.

Key words and phrases. bifurcating autoregressive process; random coefficient; weighted least squares; martingale; almost sure convergence; central limit theorem.

This paper, which is an adaptation of [4] to RCBAR processes, intends to study the asymptotic behavior of the weighted least squares (WLS) estimators of first-order RCBAR processes using a martingale approach. This martingale approach has been first proposed by Bercu et al. [3] and de Saporta et al. [6] for BAR processes. The WLS estimation of parameters branching processes was previously investigated by Wei and Winnicki [22] and Winnicki [23]. We will make use several times of the strong law of large numbers [8] as well as the central limit theorem [8, 10] for martingales, in order to investigate the asymptotic behavior of the WLS estimators. Those theorems have been previously used by Basawa and Zhou [2, 24, 25].

Several approaches appeared for BAR processes, and we tried not to set aside any of them. Thus, we took into account the classical BAR studies as seen in Huggins and Basawa [12, 13] and Huggins and Staudte [14] who studied the evolution of cell diameters and lifetimes, and also the bifurcating Markov chain model introduced by Guyon [9] and used in Delmas and Marsalle [7]. Still, we did not forget to have a look to the analogy with the Galton-Watson processes as studied in Delmas and Marsalle [7] and Heyde and Seneta [11]. Several methods have also been used for parameter estimation in RCAR processes. Koul and Schick [16] used an M-estimator while Aue et al. [1] preferred a quasi-maximum likelihood approach. Schick [19] introduced a new class of estimator that Vanecek [20] used in his work. Hwang et al. [15] also tackled the critical case where the environmental effect follows a Rademacher distribution.

The paper is organised as follows. Section 2 allows us to explain more precisely the model in which we are interested in, then Section 3 formulates the WLS estimators of the unknown parameters we will study. Section 4 permits us to introduce the martingale point of view of this paper. Section 5 collects our results concerning the asymptotic behavior of our WLS estimators, to be more accurate, we will establish the almost sure convergence, the quadratic strong law and the asymptotic normality of our estimators. Finally, the other sections gather the proofs of our main results.

2. BIFURCATING INTEGER-VALUED AUTOREGRESSIVE PROCESSES

Consider the first-order RCBAR process given, for all $n \geq 1$, by

$$(2.1) \quad \begin{cases} X_{2n} &= a_n X_n + \varepsilon_{2n} \\ X_{2n+1} &= b_n X_n + \varepsilon_{2n+1} \end{cases}$$

where the initial state X_1 is the ancestor of the process and $(\varepsilon_{2n}, \varepsilon_{2n+1})$ stands for the driven noise of the process. In all the sequel, we shall assume that $\mathbb{E}[X_1^2] < \infty$. We also assume that both $(a_n, b_n)_{n \geq 1}$ and $(\varepsilon_{2n}, \varepsilon_{2n+1})_{n \geq 1}$ are i.i.d., and that those two sequences are independent. One can see the RCBAR process given by (2.1) as a first-order random coefficient autoregressive process on a binary tree, where each node represents an individual, node 1 being the original ancestor. For all $n \geq 1$, denote the n -th generation by

$$\mathbb{G}_n = \{2^n, 2^{n+1}, \dots, 2^{n+1} - 1\}.$$

In particular, $\mathbb{G}_0 = \{1\}$ is the initial generation and $\mathbb{G}_1 = \{2, 3\}$ is the first generation of offspring from the first ancestor. Recall that the two offspring of individual n are labelled $2n$ and $2n + 1$, or conversely, the mother of individual n is $\lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ stands for the largest integer less than or equal to x . Finally denote by

$$\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$$

the sub-tree of all individuals from the original individual up to the n -th generation. One can observe that the cardinality $|\mathbb{G}_n|$ of \mathbb{G}_n is 2^n while that of \mathbb{T}_n is $|\mathbb{T}_n| = 2^{n+1} - 1$.

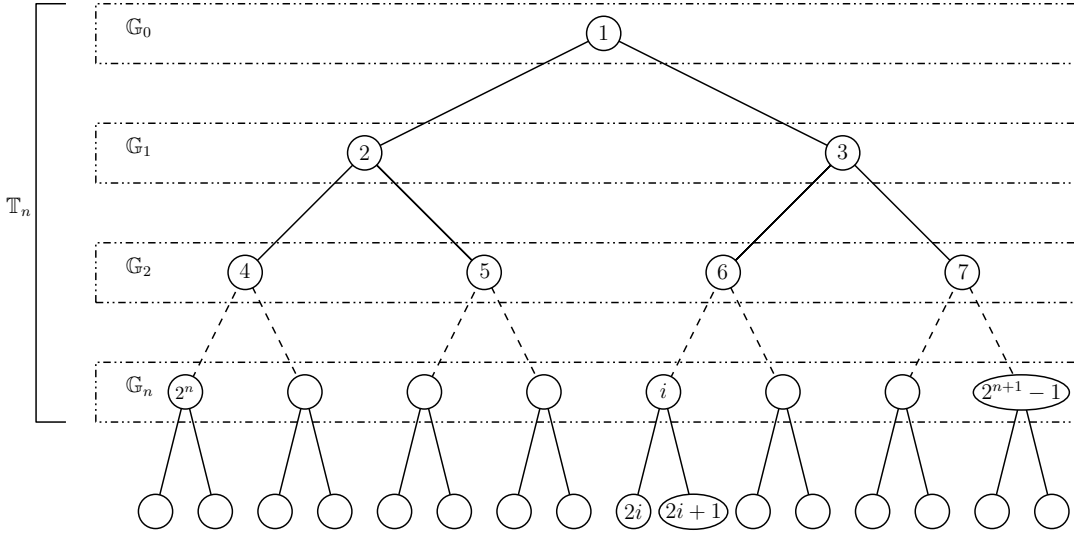


FIGURE 1. The tree associated with the RCBAR

3. WEIGHTED LEAST-SQUARES ESTIMATION

Denote by $\mathbb{F} = (\mathcal{F}_n)_{n \geq 0}$ the natural filtration associated with the first-order RCBAR process, which means that \mathcal{F}_n is the σ -algebra generated by all individuals up to the n -th generation, in other words $\mathcal{F}_n = \sigma\{X_k, k \in \mathbb{T}_n\}$. We will assume in all the sequel that, for all $n \geq 0$ and for all $k \in \mathbb{G}_n$,

$$(3.1) \quad \begin{cases} \mathbb{E}[a_k | \mathcal{F}_n] = a & \text{a.s.} \\ \mathbb{E}[b_k | \mathcal{F}_n] = b & \text{a.s.} \\ \mathbb{E}[\varepsilon_{2k} | \mathcal{F}_n] = c & \text{a.s.} \\ \mathbb{E}[\varepsilon_{2k+1} | \mathcal{F}_n] = d & \text{a.s.} \end{cases}$$

Consequently, we deduce from (2.1) and (3.1) that, for all $n \geq 0$ and for all $k \in \mathbb{G}_n$,

$$(3.2) \quad \begin{cases} X_{2k} &= aX_k + c + V_{2k}, \\ X_{2k+1} &= bX_k + d + V_{2k+1}, \end{cases}$$

where, $V_{2k} = X_{2k} - \mathbb{E}[X_{2k}|\mathcal{F}_n]$ and $V_{2k+1} = X_{2k+1} - \mathbb{E}[X_{2k+1}|\mathcal{F}_n]$. Therefore, the two relations given by (3.2) can be rewritten in a classic autoregressive form

$$(3.3) \quad \chi_n = \theta^t \Phi_n + W_n$$

where

$$\chi_n = \begin{pmatrix} X_{2n} \\ X_{2n+1} \end{pmatrix}, \quad \Phi_n = \begin{pmatrix} X_n \\ 1 \end{pmatrix}, \quad W_n = \begin{pmatrix} V_{2n} \\ V_{2n+1} \end{pmatrix},$$

and the matrix parameter

$$\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Our goal is to estimate θ from the observation of all individuals up to \mathbb{T}_n . We propose to make use of the WLS estimator $\hat{\theta}_n$ of θ which minimizes

$$\Delta_n(\theta) = \frac{1}{2} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \|\chi_k - \theta^t \Phi_k\|^2$$

where the choice of the weighting sequence $(c_n)_{n \geq 1}$ is crucial. We shall choose $c_n = 1 + X_n^2$ and we will go back to this suitable choice in Section 4. Consequently, we obviously have for all $n \geq 1$

$$(3.4) \quad \hat{\theta}_n = S_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \Phi_k \chi_k^t$$

where

$$S_n = \sum_{k \in \mathbb{T}_n} \frac{1}{c_k} \Phi_k \Phi_k^t.$$

In order to avoid useless invertibility assumption, we shall assume, without loss of generality, that for all $n \geq 0$, S_n is invertible. Otherwise, we only have to add the identity matrix of order 2, I_2 to S_n . In all what follows, we shall make a slight abuse of notation by identifying θ as well as $\hat{\theta}_n$ to

$$\text{vec}(\theta) = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \quad \text{and} \quad \text{vec}(\hat{\theta}_n) = \begin{pmatrix} \hat{a}_n \\ \hat{c}_n \\ \hat{b}_n \\ \hat{d}_n \end{pmatrix}.$$

Therefore, we deduce from (3.4) that

$$\begin{aligned} \hat{\theta}_n &= \Sigma_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \text{vec}(\Phi_k \chi_k^t), \\ &= \Sigma_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \begin{pmatrix} X_k X_{2k} \\ X_{2k} \\ X_k X_{2k+1} \\ X_{2k+1} \end{pmatrix} \end{aligned}$$

where $\Sigma_n = I_2 \otimes S_n$ and \otimes stands for the standard Kronecker product. Consequently, (3.3) yields to

$$(3.5) \quad \begin{aligned} \hat{\theta}_n - \theta &= \Sigma_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \text{vec}(\Phi_k W_k^t), \\ &= \Sigma_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \begin{pmatrix} X_k V_{2k} \\ V_{2k} \\ X_k V_{2k+1} \\ V_{2k+1} \end{pmatrix}. \end{aligned}$$

In all the sequel, we shall make use of the following moment hypotheses.

(H.1) For all $k \geq 1$,

$$\mathbb{E}[a_k^2] < 1 \quad \text{and} \quad \mathbb{E}[b_k^2] < 1.$$

(H.2) For all $n \geq 0$ and for all $k \in \mathbb{G}_n$

$$\text{Var}[a_k | \mathcal{F}_n] = \sigma_a^2 \geq 0 \quad \text{and} \quad \text{Var}[b_k | \mathcal{F}_n] = \sigma_b^2 \geq 0 \quad \text{a.s.}$$

$$\text{Var}[\varepsilon_{2k} | \mathcal{F}_n] = \sigma_c^2 > 0 \quad \text{and} \quad \text{Var}[\varepsilon_{2k+1} | \mathcal{F}_n] = \sigma_d^2 > 0 \quad \text{a.s.}$$

(H.3) For all $n \geq 0$ and for all $k, l \in \mathbb{G}_{n+1}$, if $[k/2] \neq [l/2]$, ε_k and ε_l are conditionally independent given \mathcal{F}_n and for all $k, l \in \mathbb{G}_n$, if $k \neq l$, (a_k, b_k) and (a_l, b_l) are conditionally independent given \mathcal{F}_n . While otherwise, it exists $\rho_{cd}^2 < \sigma_c^2 \sigma_d^2$ and $\rho_{ab}^2 \leq \sigma_a^2 \sigma_b^2$ such that, for all $k \in \mathbb{G}_n$

$$\mathbb{E}[(\varepsilon_{2k} - c)(\varepsilon_{2k+1} - d) | \mathcal{F}_n] = \rho_{cd} \quad \text{a.s.}$$

$$\mathbb{E}[(a_k - a)(b_k - b) | \mathcal{F}_n] = \rho_{ab} \quad \text{a.s.}$$

(H.4) One can find $\mu_a^4 \geq \sigma_a^4$, $\mu_b^4 \geq \sigma_b^4$, $\mu_c^4 > \sigma_c^4$ and $\mu_d^4 > \sigma_d^4$ such that, for all $n \geq 0$ and for all $k \in \mathbb{G}_n$

$$\mathbb{E}[(a_k - a)^4 | \mathcal{F}_n] = \mu_a^4 \quad \text{and} \quad \mathbb{E}[(b_k - b)^4 | \mathcal{F}_n] = \mu_b^4 \quad \text{a.s.}$$

$$\mathbb{E}[(\varepsilon_{2k} - c)^4 | \mathcal{F}_n] = \mu_c^4 \quad \text{and} \quad \mathbb{E}[(\varepsilon_{2k+1} - d)^4 | \mathcal{F}_n] = \mu_d^4 \quad \text{a.s.}$$

In addition, it exists $\nu_{ab}^2 \geq \rho_{ab}^2$ and $\nu_{cd}^2 > \rho_{cd}^2$ such that, for all $k \in \mathbb{G}_n$

$$\mathbb{E}[(a_k - a)^2(b_k - b)^2 | \mathcal{F}_n] = \nu_{ab}^2 \quad \text{and} \quad \mathbb{E}[(\varepsilon_{2k} - c)^2(\varepsilon_{2k+1} - d)^2 | \mathcal{F}_n] = \nu_{cd}^2 \quad \text{a.s.}$$

(H.5) It exists τ_a^6 , τ_b^6 , τ_c^6 and τ_d^6 such that

$$\mathbb{E}[(a_k - a)^6 | \mathcal{F}_n] = \tau_a^6 \quad \text{and} \quad \mathbb{E}[(b_k - b)^6 | \mathcal{F}_n] = \tau_b^6 \quad \text{a.s.}$$

$$\mathbb{E}[(\varepsilon_{2k} - c)^6 | \mathcal{F}_n] = \tau_c^6 \quad \text{and} \quad \mathbb{E}[(\varepsilon_{2k+1} - d)^6 | \mathcal{F}_n] = \tau_d^6 \quad \text{a.s.}$$

One can observe that those hypotheses allows us to consider the deterministic case where it exists some constants a, b with $\max(|a|, |b|) < 1$ such that, for all $k \geq 1$, $a_k = a$ and $b_k = b$ a.s. Moreover, under assumption **(H.2)**, we have for all $n \geq 0$ and for all $k \in \mathbb{G}_n$

$$(3.6) \quad \mathbb{E}[V_{2k}^2 | \mathcal{F}_n] = \sigma_a^2 X_k^2 + \sigma_c^2 \quad \text{and} \quad \mathbb{E}[V_{2k+1}^2 | \mathcal{F}_n] = \sigma_b^2 X_k^2 + \sigma_d^2 \quad \text{a.s.}$$

Consequently, if we choose $c_n = 1 + X_n^2$ for all $n \geq 1$, we clearly have for all $k \in \mathbb{G}_n$

$$\mathbb{E}[V_{2k}^2 | \mathcal{F}_n] \leq \max(\sigma_a^2, \sigma_c^2) c_k \quad \text{and} \quad \mathbb{E}[V_{2k+1}^2 | \mathcal{F}_n] \leq \max(\sigma_b^2, \sigma_d^2) c_k \quad \text{a.s.}$$

It is exactly the reason why we have chosen this weighting sequence into (3.4). Similar WLS estimation approach for branching processes with immigration may be found in [22] and [23]. For all $n \geq 0$ and for all $k \in \mathbb{G}_n$, denote $v_{2k} = V_{2k}^2 - \mathbb{E}[V_{2k}^2 | \mathcal{F}_n]$. We deduce from (3.6) that for all $n \geq 1$

$$V_{2n}^2 = \eta^t \psi_n + v_{2n}$$

where

$$\eta = \begin{pmatrix} \sigma_a^2 \\ \sigma_c^2 \end{pmatrix} \quad \text{and} \quad \psi_n = \begin{pmatrix} X_n^2 \\ 1 \end{pmatrix}.$$

It leads us to estimate the vector of variances η by the WLS estimator

$$(3.7) \quad \hat{\eta}_n = Q_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} \hat{V}_{2k}^2 \psi_k$$

where

$$Q_n = \sum_{k \in \mathbb{T}_n} \frac{1}{d_k} \psi_k \psi_k^t$$

and for all $k \in \mathbb{G}_n$,

$$\begin{cases} \hat{V}_{2k} &= X_{2k} - \hat{a}_n X_k - \hat{c}_n, \\ \hat{V}_{2k+1} &= X_{2k+1} - \hat{b}_n X_k - \hat{d}_n. \end{cases}$$

Finally the weighting sequence $(d_n)_{n \geq 1}$ is given, for all $n \geq 1$, by $d_n = c_n^2 = (1 + X_n^2)^2$. This choice is due to the fact that for all $n \geq 1$ and for all $k \in \mathbb{G}_n$

$$\begin{aligned} \mathbb{E}[v_{2k}^2 | \mathcal{F}_n] &= \mathbb{E}[V_{2k}^4 | \mathcal{F}_n] - (\mathbb{E}[V_{2k}^2 | \mathcal{F}_n])^2 \quad \text{a.s.} \\ &= (\mu_a^4 - \sigma_a^4) X_k^4 + 4\sigma_a^2 \sigma_c^2 X_k^2 + (\mu_c^4 - \sigma_c^4) \quad \text{a.s.} \end{aligned}$$

Consequently, as $d_n \geq 1$, we clearly have for all $n \geq 1$ and for all $k \in \mathbb{G}_n$

$$\mathbb{E}[v_{2k}^2 | \mathcal{F}_n] \leq \max(\mu_a^4 - \sigma_a^4, 2\sigma_a^2 \sigma_c^2, \mu_c^4 - \sigma_c^4) d_k \quad \text{a.s.}$$

We have a similar WLS estimator $\hat{\zeta}_n$ of the vector of variances $\zeta^t = (\sigma_b^2 \ \sigma_d^2)$ by replacing \hat{V}_{2k}^2 by \hat{V}_{2k+1}^2 into (3.7). Let us remark that, for all $n \geq 0$ and for all $k \in \mathbb{G}_n$,

$$(3.8) \quad \mathbb{E}[V_{2k} V_{2k+1} | \mathcal{F}_n] = \rho_{ab} X_n^2 + \rho_{cd}.$$

Then, for all $n \geq 0$ and for all $k \in \mathbb{G}_n$, denote $w_{2k} = V_{2k} V_{2k+1} - \mathbb{E}[V_{2k} V_{2k+1} | \mathcal{F}_n]$. We deduce from (3.8) that for all $n \geq 1$

$$V_{2k} V_{2k+1} = \nu^t \psi_k + w_{2k}$$

where $\nu^t = (\rho_{ab} \ \rho_{cd})$. It leads us to estimate the vector of covariances ν by the WLS estimator

$$(3.9) \quad \hat{\nu}_n = Q_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} \hat{V}_{2k} \hat{V}_{2k+1} \psi_k.$$

This choice is due to the fact that for all $n \geq 1$ and for all $k \in \mathbb{G}_n$

$$\mathbb{E}[V_{2k}^2 V_{2k+1}^2 | \mathcal{F}_n] = \nu_{ab}^2 X_k^4 + (\sigma_a^2 \sigma_d^2 + 4\rho_{ab}\rho_{cd} + \sigma_b^2 \sigma_c^2) X_k^2 + \nu_{cd}^2 \quad \text{a.s.}$$

Consequently, as $d_n \geq 1$, we clearly have for all $n \geq 1$ and for all $k \in \mathbb{G}_n$

$$\begin{aligned} \mathbb{E}[w_{2k}^2 | \mathcal{F}_n] &= (\nu_{ab}^2 - \rho_{ab}^2) X_k^4 + (\sigma_a^2 \sigma_d^2 + \sigma_b^2 \sigma_c^2 + 2\rho_{ab}\rho_{cd}) X_k^2 + (\nu_{cd}^2 - \rho_{cd}^2) \quad \text{a.s.} \\ &\leq \max(\nu_{ab}^2, \nu_{cd}^2, (\sigma_a^2 + \sigma_c^2)(\sigma_b^2 + \sigma_d^2)) d_k \quad \text{a.s.} \end{aligned}$$

4. A MARTINGALE APPROACH

In order to establish all the asymptotic properties of our estimators, we shall make use of a martingale approach. For all $n \geq 1$, denote

$$M_n = \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \begin{pmatrix} X_k V_{2k} \\ V_{2k} \\ X_k V_{2k+1} \\ V_{2k+1} \end{pmatrix}.$$

We can clearly rewrite (3.5) as

$$(4.1) \quad \hat{\theta}_n - \theta = \Sigma_{n-1}^{-1} M_n.$$

As in [3], we make use of the notation M_n since it appears that $(M_n)_{n \geq 1}$ a martingale. This fact is a crucial point of our study and it justifies the vector notation since most of asymptotic results for martingales were established for vector-valued martingales. Let us rewrite M_n in order to emphasize its martingale quality. Let $\Psi_n = I_2 \otimes \varphi_n$ where φ_n is the matrix of dimension 2×2^n given by

$$\varphi_n = \begin{pmatrix} \frac{X_{2^n}}{\sqrt{c_{2^n}}} & \frac{X_{2^{n+1}}}{\sqrt{c_{2^{n+1}}}} & \cdots & \frac{X_{2^{n+1}-1}}{\sqrt{c_{2^{n+1}-1}}} \\ 1 & 1 & \cdots & 1 \\ \frac{1}{\sqrt{c_{2^n}}} & \frac{1}{\sqrt{c_{2^{n+1}}}} & \cdots & \frac{1}{\sqrt{c_{2^{n+1}-1}}} \end{pmatrix}.$$

It represents the individuals of the n -th generation which is also the collection of all $\Phi_k / \sqrt{c_k}$ where k belongs to \mathbb{G}_n . Let ξ_n be the random vector of dimension 2^n

$$\xi_n^t = \left(\frac{V_{2^n}}{\sqrt{c_{2^{n-1}}}} \quad \frac{V_{2^{n+2}}}{\sqrt{c_{2^{n-1}+1}}} \quad \cdots \quad \frac{V_{2^{n+1}-2}}{\sqrt{c_{2^{n-1}}}} \quad \frac{V_{2^{n+1}}}{\sqrt{c_{2^{n-1}}}} \quad \frac{V_{2^{n+3}}}{\sqrt{c_{2^{n-1}+1}}} \quad \cdots \quad \frac{V_{2^{n+1}-1}}{\sqrt{c_{2^{n-1}}}} \right).$$

The vector ξ_n gathers the noise variables of \mathbb{G}_n . The special ordering separating odd and even indices has been made in [3] so that M_n can be written as

$$M_n = \sum_{k=1}^n \Psi_{k-1} \xi_k.$$

Under (3.1), we clearly have for all $n \geq 0$, $\mathbb{E}[\xi_{n+1} | \mathcal{F}_n] = 0$ a.s. and Ψ_n is \mathcal{F}_n -measurable. In addition it is not hard to see that under **(H.1)** to **(H.2)**, (M_n) is

a locally square integrable vector martingale with increasing process given, for all $n \geq 1$, by

$$(4.2) \quad \langle M \rangle_n = \sum_{k=0}^{n-1} \Psi_k \mathbb{E}[\xi_{k+1} \xi_{k+1}^t | \mathcal{F}_k] \Psi_k^t = \sum_{k=0}^{n-1} L_k \quad \text{a.s.}$$

where

$$(4.3) \quad L_k = \sum_{i \in \mathbb{G}_k} \frac{1}{c_i^2} \begin{pmatrix} P(X_i) & Q(X_i) \\ Q(X_i) & R(X_i) \end{pmatrix} \otimes \begin{pmatrix} X_i^2 & X_i \\ X_i & 1 \end{pmatrix}.$$

with

$$\begin{cases} P(X) = \sigma_a^2 X^2 + \sigma_c^2, \\ Q(X) = \rho_{ab} X^2 + \rho_{cd}, \\ R(X) = \sigma_b^2 X^2 + \sigma_d^2. \end{cases}$$

One can remark that we obviously have $\langle M \rangle_n = \mathcal{O}(n)$ but it is necessary to establish the convergence of $\langle M \rangle_n$, properly normalized, in order to prove the asymptotic results for our RCBAR estimators $\hat{\theta}_n$, $\hat{\eta}_n$, $\hat{\zeta}_n$ and $\hat{\nu}_n$.

5. MAIN RESULTS

We have to introduce some more notations in order to state our main results. From the original process $(X_n)_{n \geq 1}$, we shall define a new process $(Y_n)_{n \geq 1}$ recursively defined by $Y_1 = X_1$, and if $Y_n = X_k$ with $n, k \geq 1$, then

$$Y_{n+1} = X_{2k+\kappa_n}$$

where $(\kappa_n)_{n \geq 1}$ is a sequence of i.i.d. random variables with Bernoulli $\mathcal{B}(1/2)$ distribution. Such a construction may be found in [9] for the asymptotic analysis of BAR processes. The process (Y_n) gathers the values of the original process (X_n) along the random branch of the binary tree (\mathbb{T}_n) given by (κ_n) . Denote by k_n the unique $k \geq 1$ such that $Y_n = X_k$. Then, for all $n \geq 1$, we have

$$(5.1) \quad Y_{n+1} = \tilde{a}_{n+1} Y_n + e_{n+1}$$

where, with k_n the unique number k such that $Y_n = X_k$,

$$(5.2) \quad \tilde{a}_{n+1} = \begin{cases} a_{k_n} & \text{if } \kappa_n = 0, \\ b_{k_n} & \text{otherwise,} \end{cases} \quad \text{and} \quad e_n = \varepsilon_{k_n}.$$

Lemma 5.1. *Assume that (H.1) and (H.2) are satisfied. Then, we have*

$$Y_n \xrightarrow{\mathcal{L}} T$$

where T is a positive non degenerate random variable with $\mathbb{E}[T^2] < \infty$.

Denote $\mathcal{C}_b^1(\mathbb{R}_+) = \left\{ f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \mid \exists \gamma > 0, \forall x \geq 0, (|f'(x)| + |f(x)|) \leq \gamma \right\}$.

Lemma 5.2. *Assume that (H.1) and (H.2) are satisfied. Then, for all $f \in \mathcal{C}_b^1(\mathbb{R}_+)$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} f(X_k) = \mathbb{E}[f(T)] \quad a.s.$$

Proposition 5.3. *Assume that (H.1) to (H.3) are satisfied. Then, we have*

$$(5.3) \quad \lim_{n \rightarrow \infty} \frac{\langle M \rangle_n}{|\mathbb{T}_{n-1}|} = L \quad a.s.$$

where L is the positive definite matrix given by

$$L = \mathbb{E} \left[\frac{1}{(1+T^2)^2} \begin{pmatrix} P(T) & Q(T) \\ Q(T) & R(T) \end{pmatrix} \otimes \begin{pmatrix} T^2 & T \\ T & 1 \end{pmatrix} \right].$$

Our first result deals with the almost sure convergence of our WLS estimator $\hat{\theta}_n$.

Theorem 5.4. *Assume that (H.1) to (H.4) satisfied. Then, $\hat{\theta}_n$ converges almost surely to θ with the rate of convergence*

$$\|\hat{\theta}_n - \theta\|^2 = \mathcal{O} \left(\frac{n}{|\mathbb{T}_{n-1}|} \right) \quad a.s.$$

In addition, we also have the quadratic strong law

$$(5.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\mathbb{T}_{k-1}| (\hat{\theta}_k - \theta)^t \Lambda (\hat{\theta}_k - \theta) = \text{tr}(\Lambda^{-1/2} L \Lambda^{-1/2}) \quad a.s.$$

where

$$(5.5) \quad \Lambda = I_2 \otimes C \quad \text{and} \quad C = \mathbb{E} \left[\frac{1}{1+T^2} \begin{pmatrix} T^2 & T \\ T & 1 \end{pmatrix} \right].$$

Our second result concerns the almost sure asymptotic properties of our WLS variance and covariance estimators $\hat{\eta}_n$, $\hat{\zeta}_n$ and $\hat{\nu}_n$. Let

$$\begin{aligned} \eta_n &= Q_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} V_{2k}^2 \psi_k, \\ \zeta_n &= Q_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} V_{2k+1}^2 \psi_k, \\ \nu_n &= Q_{n-1}^{-1} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} V_{2k} V_{2k+1} \psi_k. \end{aligned}$$

Theorem 5.5. *Assume that (H.1) to (H.4) are satisfied. Then, $\hat{\eta}_n$ and $\hat{\zeta}_n$ converge almost surely to η and ζ respectively. More precisely,*

$$(5.6) \quad \|\hat{\eta}_n - \eta_n\| = \mathcal{O} \left(\frac{n}{|\mathbb{T}_{n-1}|} \right) \quad a.s.$$

$$(5.7) \quad \|\hat{\zeta}_n - \zeta_n\| = \mathcal{O} \left(\frac{n}{|\mathbb{T}_{n-1}|} \right) \quad a.s.$$

In addition, $\widehat{\nu}_n$ converges almost surely to ν with

$$(5.8) \quad \|\widehat{\nu}_n - \nu_n\| = \mathcal{O}\left(\frac{n}{|\mathbb{T}_{n-1}|}\right) \quad a.s.$$

Remark 5.6. We also have the almost sure rates of convergence

$$\|\widehat{\eta}_n - \eta\|^2 = \mathcal{O}\left(\frac{n}{|\mathbb{T}_{n-1}|}\right), \quad \|\widehat{\zeta}_n - \zeta\|^2 = \mathcal{O}\left(\frac{n}{|\mathbb{T}_{n-1}|}\right), \quad \|\widehat{\nu}_n - \nu\|^2 = \mathcal{O}\left(\frac{n}{|\mathbb{T}_{n-1}|}\right) \quad a.s.$$

Our last result is devoted to the asymptotic normality of our WLS estimators $\widehat{\theta}_n$, $\widehat{\eta}_n$, $\widehat{\zeta}_n$ and $\widehat{\nu}_n$.

Theorem 5.7. Assume that (H.1) to (H.5) are satisfied. Then, we have the asymptotic normality

$$(5.9) \quad \sqrt{|\mathbb{T}_{n-1}|}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Lambda^{-1} L \Lambda^{-1}).$$

In addition, we also have

$$(5.10) \quad \sqrt{|\mathbb{T}_{n-1}|}(\widehat{\eta}_n - \eta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1} M_{ac} D^{-1}),$$

$$(5.11) \quad \sqrt{|\mathbb{T}_{n-1}|}(\widehat{\zeta}_n - \zeta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1} M_{bd} D^{-1}),$$

where

$$D = \mathbb{E} \left[\frac{1}{(1 + T^2)^2} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right],$$

$$M_{ac} = \mathbb{E} \left[\frac{(\mu_a^4 - \sigma_a^4)T^4 + 4\sigma_a^2\sigma_c^2T^2 + (\mu_c^4 - \sigma_c^4)}{(1 + T^2)^4} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right],$$

$$M_{bd} = \mathbb{E} \left[\frac{(\mu_b^4 - \sigma_b^4)T^4 + 4\sigma_b^2\sigma_d^2T^2 + (\mu_d^4 - \sigma_d^4)}{(1 + T^2)^4} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right].$$

Finally,

$$(5.12) \quad \sqrt{|\mathbb{T}_{n-1}|}(\widehat{\nu}_n - \nu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1} H D^{-1})$$

where

$$H = \mathbb{E} \left[\frac{(\nu_{ab}^2 - \rho_{ab}^2)T^4 + (\sigma_a^2\sigma_d^2 + \sigma_b^2\sigma_c^2 + 2\rho_{ab}\rho_{cd})T^2 + (\nu_{cd}^2 - \rho_{cd}^2)}{(1 + T^2)^4} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right].$$

The rest of the paper is dedicated to the proof of our main results.

6. PROOF OF LEMMA 5.1

We can reformulate (5.1) and (5.2) as

$$Y_n = \widetilde{a}_n \widetilde{a}_{n-1} \dots \widetilde{a}_2 Y_1 + \sum_{k=2}^{n-1} \widetilde{a}_n \widetilde{a}_{n-1} \dots \widetilde{a}_{k+1} e_k + e_n.$$

We already made the assumption that both $(a_n, b_n)_{n \geq 1}$ and $(\varepsilon_{2n}, \varepsilon_{2n+1})_{n \geq 1}$ are i.i.d. and that those two sequences are independent. Consequently, the couples (\widetilde{a}_k, e_k)

and $(\tilde{a}_{n-k+2}, e_{n-k+1})$ share the same distribution. Hence, for all $n \geq 2$, Y_n has the same distribution than the random variable

$$\begin{aligned} Z_n &= \tilde{a}_2 \dots \tilde{a}_n Y_1 + \sum_{k=2}^{n-1} \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{n-k+1} e_{n-k+2} + e_2, \\ &= \tilde{a}_2 \dots \tilde{a}_n Y_1 + \sum_{k=3}^n \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{k-1} e_k + e_2. \end{aligned}$$

For the sake of simplicity, we will denote

$$(6.1) \quad Z_n = \tilde{a}_2 \dots \tilde{a}_n Y_1 + \sum_{k=2}^n \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{k-1} e_k.$$

On the first hand,

$$\mathbb{E}[\tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_n Y_1] = \mathbb{E}[\tilde{a}_2]^{n-1} \mathbb{E}[Y_1]$$

and since

$$|\mathbb{E}[\tilde{a}_2]| = \left| \frac{a+b}{2} \right| < 1$$

this immediately leads to

$$\lim_{n \rightarrow \infty} \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_n Y_1 = 0 \quad \text{a.s.}$$

On the other hand, let T_n be defined as

$$T_n = \sum_{k=2}^n \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{k-1} e_k$$

and T given by

$$T = \sum_{k=2}^{\infty} \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{k-1} e_k.$$

We have

$$\begin{aligned} \mathbb{E}[|T - T_n|] &= \mathbb{E} \left[\left| \sum_{k=n+1}^{\infty} \tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{k-1} e_k \right| \right], \\ &\leq \sum_{k=n+1}^{\infty} \mathbb{E} [|\tilde{a}_2 \tilde{a}_3 \dots \tilde{a}_{k-1} e_k|], \\ &\leq \mathbb{E}[|e_2|] \sum_{k=n+1}^{\infty} \mathbb{E} [|\tilde{a}_2|]^{k-2}. \end{aligned}$$

In addition, $\mathbb{E}[a_n^2] < 1$ and $\mathbb{E}[b_n^2] < 1$ which leads to $\mathbb{E}[\tilde{a}_n^2] < 1$ and $\mathbb{E}[|\tilde{a}_n|] < 1$. Consequently,

$$\mathbb{E}[|T - T_n|] \leq \mathbb{E} [|\tilde{a}_2|]^{n-1} \frac{\mathbb{E}[|e_2|]}{1 - \mathbb{E} [|\tilde{a}_2|]}.$$

This proves that $T_n \xrightarrow{L^1} T$ which immediately implies that

$$T_n \xrightarrow{\mathcal{L}} T \quad \text{and} \quad Y_n \xrightarrow{\mathcal{L}} T.$$

Moreover, we can easily see that **(H.1)** allows us to say that $\mathbb{E}[T^2] < \infty$ thanks to Cauchy-Schwarz inequality. It only remains to prove that T is not degenerate. Let us write T_n as

$$T_n = e_2 + \sum_{k=3}^n \tilde{a}_2 \dots \tilde{a}_{k-1} e_k.$$

Since e_2 is independent of $(\tilde{a}_2 \dots \tilde{a}_{k-1} e_k)_{k \geq 3}$, we have

$$(6.2) \quad \text{Var}(T_n) = \text{Var}(e_2) + \text{Var}\left(\sum_{k=3}^n \tilde{a}_2 \dots \tilde{a}_{k-1} e_k\right) \geq \text{Var}(e_2).$$

Moreover, it is easy to see that

$$(6.3) \quad \text{Var}(e_2) = \frac{\sigma_c^2 + c^2 + \sigma_d^2 + d^2}{2} - \left(\frac{c+d}{2}\right)^2 = \frac{\sigma_c^2 + \sigma_d^2}{2} + \frac{(c-d)^2}{4} \geq \frac{\sigma_c^2 + \sigma_d^2}{2} > 0.$$

Consequently, as (T_n) converges in distribution to T , we obtain from Levy's continuity theorem, (6.2) and (6.3) that

$$\text{Var}(T) = \lim_{n \rightarrow \infty} \text{Var}(T_n) \geq \text{Var}(e_2) > 0.$$

7. PROOF OF LEMMA 5.2

We shall now prove that for all $f \in \mathcal{C}_b^1(\mathbb{R}_+)$,

$$\lim_{n \rightarrow \infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} f(X_k) = \mathbb{E}[f(T)].$$

Denote $g = f - \mathbb{E}[f(T)]$,

$$\overline{M}_{\mathbb{T}_n}(f) = \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} f(X_k) \quad \text{and} \quad \overline{M}_{\mathbb{G}_n}(f) = \frac{1}{|\mathbb{G}_n|} \sum_{k \in \mathbb{G}_n} f(X_k).$$

Via Lemma A.2 of [3], it is only necessary to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{|\mathbb{G}_n|} \sum_{k \in \mathbb{G}_n} g(X_k) = 0 \quad \text{a.s.}$$

We shall follow the induced Markov chain approach, originally proposed by Guyon in [9]. Let Q be the transition probability of (Y_n) , Q^p the p -th iterated of Q . In addition, denote by ν the distribution of $Y_1 = X_1$ and νQ^p the law of Y_p . Finally, let P be the transition probability of (X_n) as defined in [9]. We obtain from relation (7) of [9] that for all $n \geq 0$

$$\mathbb{E}[\overline{M}_{\mathbb{G}_n}(g)^2] = \frac{1}{2^n} \nu Q^n g^2 + \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \nu Q^k P(Q^{n-k-1} g \star Q^{n-k-1} g)$$

where, for all $x, y \in \mathbb{N}$, $(f \star g)(x, y) = f(x)g(y)$. Consequently,

$$(7.1) \quad \begin{aligned} \sum_{n=0}^{\infty} \mathbb{E}[\overline{M}_{\mathbb{G}_n}(g)^2] &= \sum_{n=0}^{\infty} \frac{1}{2^n} \nu Q^n g^2 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \nu Q^k P(Q^{n-k-1} g \star Q^{n-k-1} g), \\ &\leq \sum_{k=0}^{\infty} \frac{1}{2^k} \nu Q^k \left(g^2 + P \left(\sum_{l=0}^{\infty} |Q^l g \star Q^l g| \right) \right). \end{aligned}$$

However, for all $x \in \mathbb{N}$,

$$Q^n g(x) = Q^n f(x) - \mathbb{E}[f(T)] = \mathbb{E}_x[f(Y_n) - f(T)] = \mathbb{E}_x[f(Z_n) - f(T)]$$

where Z_n is given by (6.1). Hence, we deduce from the mean value theorem and Cauchy-Schwarz inequality that

$$(7.2) \quad |Q^n g(x)| \leq \mathbb{E}_x[W_n |Z_n - T|] \leq \mathbb{E}_x[W_n^2]^{1/2} \mathbb{E}_x[(Z_n - T)^2]^{1/2}$$

where

$$W_n = \sup_{z \in [Z_n, T]} |f'(z)|.$$

By the very definition of $\mathcal{C}_b^1(\mathbb{R}_+)$, one can find some constant $\gamma > 0$ such that $|f'(z)| \leq \gamma$. Hence,

$$(7.3) \quad \mathbb{E}_x[W_n^2]^{1/2} \leq \gamma.$$

Furthermore

$$Z_n - T = \tilde{a}_2 \dots \tilde{a}_n Y_1 - \sum_{k=n}^{\infty} \tilde{a}_2 \dots \tilde{a}_k e_{k+1}$$

and the triangle inequality allows us to say that

$$(7.4) \quad \begin{aligned} \mathbb{E}_x[(Z_n - T)^2]^{1/2} &\leq \mathbb{E}_x[(\tilde{a}_2 \dots \tilde{a}_n Y_1)^2]^{1/2} + \sum_{k=n}^{\infty} \mathbb{E}_x[(\tilde{a}_2 \dots \tilde{a}_k e_{k+1})^2]^{1/2} \\ &\leq \mathbb{E}[\tilde{a}_2^2]^{(n-1)/2} \mathbb{E}_x[Y_1^2]^{1/2} + \sum_{k=n}^{\infty} \mathbb{E}_x[\tilde{a}_2^2]^{(k-1)/2} \mathbb{E}[e_{k+1}^2]^{1/2} \\ &\leq \sqrt{\mathbb{E}[\tilde{a}_2^2]}^{n-1} \left(|x| + \frac{\mathbb{E}[e_2^2]^{1/2}}{1 - \mathbb{E}[\tilde{a}_2^2]^{1/2}} \right) \\ &\leq \alpha \sqrt{\mathbb{E}[\tilde{a}_2^2]}^n (1 + |x|) \end{aligned}$$

where

$$\alpha = \max \left(1, \frac{\mathbb{E}[e_2^2]^{1/2}}{1 - \mathbb{E}[\tilde{a}_2^2]^{1/2}} \right).$$

Finally, we obtain from (7.2) together with (7.3) and (7.4) that

$$|Q^n g(x)| \leq \gamma \alpha \sqrt{\mathbb{E}[\tilde{a}_2^2]}^{n-1} (1 + |x|).$$

Therefore,

$$(7.5) \quad P \left(\sum_{n=0}^{\infty} |Q^n g \star Q^n g| \right) \leq \frac{\gamma^2 \alpha^2}{1 - \mathbb{E}[\tilde{a}_2^2]} P(h \star h)$$

where, for all $x \in \mathbb{N}$, $h(x) = 1 + |x|$. We are now in position to prove that

$$(7.6) \quad \mathbb{E} \left[\sum_{n=0}^{\infty} \overline{M}_{\mathbb{G}_n}(g)^2 \right] < \infty.$$

Let G be the random vector defined by $G(x) = (a_1 x + \varepsilon_2, b_1 x + \varepsilon_3)^t$. We can easily see from **(H.2)** that it exists some constant $\beta > 0$ such that

$$P(h \star h)(x) = \mathbb{E}[(h \star h)(G(x))] \leq \beta(1 + x^2).$$

Consequently, since, for all $z \in \mathbb{R}$, $|g(z)| \leq 2\gamma$, we obtain from (7.1) together with (7.5) that

$$(7.7) \quad \begin{aligned} \sum_{n=0}^{\infty} \mathbb{E}[\overline{M}_{\mathbb{G}_n}(g)^2] &\leq \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\mathbb{E}[g^2(Y_k)] + \frac{\beta \gamma^2 \alpha^2}{1 - \mathbb{E}[\tilde{a}_2^2]} (1 + \mathbb{E}[Y_k^2]) \right), \\ &\leq \left(8\gamma^2 + \frac{\beta \gamma^2 \alpha^2}{1 - \mathbb{E}[\tilde{a}_2^2]} \right) \left(1 + \sum_{k=0}^{\infty} \frac{1}{2^k} \mathbb{E}[Y_k^2] \right). \end{aligned}$$

In addition, we also have

$$(7.8) \quad \begin{aligned} \mathbb{E}[Y_k^2]^{1/2} &= \mathbb{E}[Z_k^2]^{1/2}, \\ &\leq \mathbb{E}_x[(\tilde{a}_2 \dots \tilde{a}_n Y_1)^2]^{1/2} + \sum_{k=2}^n \mathbb{E}_x[(\tilde{a}_2 \dots \tilde{a}_{k-1} e_k)^2]^{1/2}, \\ &\leq \mathbb{E}[\tilde{a}_2^2]^{(n-1)/2} \mathbb{E}_x[Y_1^2]^{1/2} + \sum_{k=2}^{\infty} \mathbb{E}_x[\tilde{a}_2^2]^{(k-2)/2} \mathbb{E}[e_{k+1}^2]^{1/2}, \\ &\leq \mathbb{E}[X_1^2]^{1/2} + \frac{\mathbb{E}[e_2^2]^{1/2}}{1 - \mathbb{E}[\tilde{a}_2^2]^{1/2}}. \end{aligned}$$

Then, (7.7) and (7.8) immediately lead to (7.6). Finally, the monotone convergence theorem implies that

$$\lim_{n \rightarrow \infty} \overline{M}_{\mathbb{G}_n}(g) = 0 \quad \text{a.s.}$$

which completes the proof of Lemma 5.2.

8. PROOF OF PROPOSITION 5.3

The almost sure convergence (5.3) immediately follows from (4.2) and (4.3) together with Lemma 5.2. It only remains to prove that $\det(L) > 0$ where the limiting matrix L can be rewritten as

$$L = \mathbb{E}[\Gamma \otimes \mathcal{C}]$$

where

$$\Gamma = \begin{pmatrix} P(T) & Q(T) \\ Q(T) & R(T) \end{pmatrix} \quad \text{and} \quad \mathcal{C} = \frac{1}{(1 + T^2)^2} \begin{pmatrix} T^2 & T \\ T & 1 \end{pmatrix}.$$

We have

$$(8.1) \quad \begin{aligned} L &= \mathbb{E} \left[\begin{pmatrix} \sigma_a^2 T^2 & \rho_{ab} T^2 \\ \rho_{ab} T^2 & \sigma_b^2 T^2 \end{pmatrix} \otimes \mathcal{C} \right] + \mathbb{E} \left[\begin{pmatrix} \sigma_c^2 & \rho_{cd} \\ \rho_{cd} & \sigma_d^2 \end{pmatrix} \otimes \mathcal{C} \right], \\ &= \begin{pmatrix} \sigma_a^2 & \rho_{ab} \\ \rho_{ab} & \sigma_b^2 \end{pmatrix} \otimes \mathbb{E}[T^2 \mathcal{C}] + \begin{pmatrix} \sigma_c^2 & \rho_{cd} \\ \rho_{cd} & \sigma_d^2 \end{pmatrix} \otimes \mathbb{E}[\mathcal{C}]. \end{aligned}$$

We shall prove that $\mathbb{E}[\mathcal{C}]$ is a positive definite matrix and that $\mathbb{E}[T^2 \mathcal{C}]$ is a positive matrix. Denote by λ_1 and λ_2 the two eigenvalues of the real symmetric matrix $\mathbb{E}[\mathcal{C}]$. We clearly have

$$\lambda_1 + \lambda_2 = \text{tr}(\mathbb{E}[\mathcal{C}]) = \mathbb{E} \left[\frac{T^2 + 1}{(1 + T^2)^2} \right] > 0$$

and

$$\lambda_1 \lambda_2 = \det(\mathbb{E}[\mathcal{C}]) = \mathbb{E} \left[\frac{T^2}{(1 + T^2)^2} \right] \mathbb{E} \left[\frac{1}{(1 + T^2)^2} \right] - \mathbb{E} \left[\frac{T}{(1 + T^2)^2} \right]^2 \geq 0$$

thanks to the Cauchy-Schwarz inequality and $\lambda_1 \lambda_2 = 0$ if and only if T is degenerate, which is not the case thanks to Lemma 5.1. Consequently, $\mathbb{E}[\mathcal{C}]$ is a positive definite matrix. In the same way, we can prove that $\mathbb{E}[T\mathcal{C}]$ is a positive matrix. Since the Kronecker product of two positive (respectively definite positive) matrices is a positive (respectively definite positive) matrix, we deduce from (8.1) that L is positive definite as soon as $\rho_{cd}^2 < \sigma_c^2 \sigma_d^2$ and $\rho_{ab}^2 \leq \sigma_a^2 \sigma_b^2$ which is the case thanks to **(H.3)**.

9. PROOF OF THEOREM 5.4

We will follow the same approach as in Bercu et al. [3]. For all $n \geq 1$, let $\mathcal{V}_n = M_n^t \Sigma_{n-1}^{-1} M_n = (\hat{\theta}_n - \theta)^t \Sigma_{n-1} (\hat{\theta}_n - \theta)$. First of all, we have

$$\begin{aligned} \mathcal{V}_{n+1} &= M_{n+1}^t \Sigma_n^{-1} M_{n+1} = (M_n + \Delta M_{n+1})^t \Sigma_n^{-1} (M_n + \Delta M_{n+1}), \\ &= M_n^t \Sigma_n^{-1} M_n + 2M_n^t \Sigma_n^{-1} \Delta M_{n+1} + \Delta M_{n+1}^t \Sigma_n^{-1} \Delta M_{n+1}, \\ &= \mathcal{V}_n - M_n^t (\Sigma_{n-1}^{-1} - \Sigma_n^{-1}) M_n + 2M_n^t \Sigma_n^{-1} \Delta M_{n+1} + \Delta M_{n+1}^t \Sigma_n^{-1} \Delta M_{n+1}. \end{aligned}$$

By summing over this identity, we obtain the main decomposition

$$(9.1) \quad \mathcal{V}_{n+1} + \mathcal{A}_n = \mathcal{V}_1 + \mathcal{B}_{n+1} + \mathcal{W}_{n+1}$$

where

$$\begin{aligned} \mathcal{A}_n &= \sum_{k=1}^n M_k^t (\Sigma_{k-1}^{-1} - \Sigma_k^{-1}) M_k, \\ \mathcal{B}_{n+1} &= 2 \sum_{k=1}^n M_k^t \Sigma_k^{-1} \Delta M_{k+1} \quad \text{and} \quad \mathcal{W}_{n+1} = \sum_{k=1}^n \Delta M_{k+1}^t \Sigma_k^{-1} \Delta M_{k+1}. \end{aligned}$$

Lemma 9.1. *Assume that **(H.1)** to **(H.3)** are satisfied. Then, we have*

$$(9.2) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{W}_n}{n} = \frac{1}{2} \text{tr}((I_2 \otimes C)^{-1/2} L (I_2 \otimes C)^{-1/2}) \quad a.s.$$

where C is the positive matrix given by (5.5). In addition, we also have

$$(9.3) \quad \mathcal{B}_{n+1} = o(n) \quad a.s.$$

and

$$(9.4) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{V}_{n+1} + \mathcal{A}_n}{n} = \frac{1}{2} \text{tr}((I_2 \otimes C)^{-1/2} L (I_2 \otimes C)^{-1/2}) \quad a.s.$$

Proof. First of all, we have $\mathcal{W}_{n+1} = \mathcal{T}_{n+1} + \mathcal{R}_{n+1}$ where

$$\begin{aligned} \mathcal{T}_{n+1} &= \sum_{k=1}^n \frac{\Delta M_{k+1}^t (I_2 \otimes C)^{-1} \Delta M_{k+1}}{|\mathbb{T}_k|}, \\ \mathcal{R}_{n+1} &= \sum_{k=1}^n \frac{\Delta M_{k+1}^t (|\mathbb{T}_k| \Sigma_k^{-1} - (I_2 \otimes C)^{-1}) \Delta M_{k+1}}{|\mathbb{T}_k|}. \end{aligned}$$

One can observe that $\mathcal{T}_{n+1} = \text{tr}((I_2 \otimes C)^{-1/2} \mathcal{H}_{n+1} (I_2 \otimes C)^{-1/2})$ where

$$\mathcal{H}_{n+1} = \sum_{k=1}^n \frac{\Delta M_{k+1} \Delta M_{k+1}^t}{|\mathbb{T}_k|}.$$

Our aim is to make use of the strong law of large numbers for martingale transforms, so we start by adding and subtracting a term involving the conditional expectation of $\Delta \mathcal{H}_{n+1}$ given \mathcal{F}_n . We have thanks to relation (4.3) that for all $n \geq 0$, $\mathbb{E}[\Delta M_{n+1} \Delta M_{n+1}^t | \mathcal{F}_n] = L_n$. Consequently, we can split \mathcal{H}_{n+1} into two terms

$$\mathcal{H}_{n+1} = \sum_{k=1}^n \frac{L_k}{|\mathbb{T}_k|} + \mathcal{K}_{n+1},$$

where

$$\mathcal{K}_{n+1} = \sum_{k=1}^n \frac{\Delta M_{k+1} \Delta M_{k+1}^t - L_k}{|\mathbb{T}_k|}.$$

It clearly follows from convergence (5.3) that

$$\lim_{n \rightarrow \infty} \frac{L_n}{|\mathbb{T}_n|} = \frac{1}{2} L \quad a.s.$$

Hence, Cesaro convergence immediately implies that

$$(9.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{L_k}{|\mathbb{T}_k|} = \frac{1}{2} L \quad a.s.$$

On the other hand, the sequence $(\mathcal{K}_n)_{n \geq 2}$ is obviously a square integrable martingale. Moreover, we have

$$\Delta \mathcal{K}_{n+1} = \mathcal{K}_{n+1} - \mathcal{K}_n = \frac{1}{|\mathbb{T}_n|} (\Delta M_{n+1} \Delta M_{n+1}^t - L_n).$$

For all $u \in \mathbb{R}^4$, denote $\mathcal{K}_n(u) = u^t \mathcal{K}_n u$. It follows from tedious but straightforward calculations, together with Lemma 5.2, that the increasing process of the martingale $(\mathcal{K}_n(u))_{n \geq 2}$ satisfies $\langle \mathcal{K}(u) \rangle_n = \mathcal{O}(n)$ a.s. Therefore, we deduce from the strong

law of large numbers for martingales that for all $u \in \mathbb{R}^4$, $\mathcal{K}_n(u) = o(n)$ a.s. leading to $\mathcal{K}_n = o(n)$ a.s. Hence, we infer from (9.5) that

$$(9.6) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{H}_{n+1}}{n} = \frac{1}{2}L \quad \text{a.s.}$$

Via the same arguments as in the proof of convergence (5.3), we find that

$$(9.7) \quad \lim_{n \rightarrow \infty} \frac{\Sigma_n}{|\mathbb{T}_n|} = I_2 \otimes C \quad \text{a.s.}$$

where C is the positive definite matrix given by (5.5). Then, we obtain from (9.6) that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{T}_n}{n} = \frac{1}{2} \text{tr}((I_2 \otimes C)^{-1/2} L (I_2 \otimes C)^{-1/2}) \quad \text{a.s.}$$

which allows us to say that $\mathcal{R}_n = o(n)$ a.s. leading to (9.2) We are now in position to prove (9.3). Let us recall that

$$\mathcal{B}_{n+1} = 2 \sum_{k=1}^n M_k^t \Sigma_k^{-1} \Delta M_{k+1} = 2 \sum_{k=1}^n M_k^t \Sigma_k^{-1} \Psi_k \xi_{k+1}.$$

Hence, $(\mathcal{B}_n)_{n \geq 2}$ is a square integrable martingale. In addition, we have

$$\Delta \mathcal{B}_{n+1} = 2 M_n^t \Sigma_n^{-1} \Delta M_{n+1}.$$

Thus

$$\begin{aligned} \mathbb{E}[(\Delta \mathcal{B}_{n+1})^2 | \mathcal{F}_n] &= 4 \mathbb{E}[M_n^t \Sigma_n^{-1} \Delta M_{n+1} \Delta M_{n+1}^t \Sigma_n^{-1} M_n | \mathcal{F}_n] \quad \text{a.s.} \\ &= 4 M_n^t \Sigma_n^{-1} \mathbb{E}[\Delta M_{n+1} \Delta M_{n+1}^t | \mathcal{F}_n] \Sigma_n^{-1} M_n \quad \text{a.s.} \\ &= 4 M_n^t \Sigma_n^{-1} L_n \Sigma_n^{-1} M_n \quad \text{a.s.} \end{aligned}$$

We can observe that

$$L_n = \sum_{k \in \mathbb{G}_n} \frac{1}{c_k^2} \begin{pmatrix} P(X_k) & Q(X_k) \\ Q(X_k) & R(X_k) \end{pmatrix} \otimes \begin{pmatrix} X_k^2 & X_k \\ X_k & 1 \end{pmatrix}$$

and

$$\Psi_n \Psi_n^t = \sum_{k \in \mathbb{G}_n} \frac{1}{c_k} I_2 \otimes \begin{pmatrix} X_k^2 & X_k \\ X_k & 1 \end{pmatrix}.$$

For $\alpha = \max(\sigma_a^2, \sigma_c^2) + \max(\sigma_b^2, \sigma_d^2) + \max(|\rho_{ab}|, |\rho_{cd}|)$, denote

$$\Delta_n = \begin{pmatrix} \alpha - \frac{P(X_n)}{c_n} & -\frac{Q(X_n)}{c_n} \\ -\frac{Q(X_n)}{c_n} & \alpha - \frac{R(X_n)}{c_n} \end{pmatrix}.$$

We can rewrite $\alpha \Psi_n \Psi_n^t - L_n$ as

$$\alpha \Psi_n \Psi_n^t - L_n = \sum_{k \in \mathbb{G}_n} \frac{1}{c_k} \Delta_k \otimes \begin{pmatrix} X_k^2 & X_k \\ X_k & 1 \end{pmatrix}.$$

It is not hard to see that Δ_n is a positive definite matrix. As a matter of fact, we deduce from the elementary inequalities

$$(9.8) \quad \begin{cases} 0 < P(X) \leq \max(\sigma_a^2, \sigma_c^2)(1 + X^2), \\ 0 < R(X) \leq \max(\sigma_b^2, \sigma_d^2)(1 + X^2), \\ |Q(X)| \leq \max(|\rho_{ab}|, |\rho_{cd}|)(1 + X^2), \end{cases}$$

that

$$\text{tr}(\Delta_n) = 2\alpha - \frac{P(X_n)}{c_n} - \frac{R(X_n)}{c_n} \geq 2\alpha - \max(\sigma_a^2, \sigma_c^2) - \max(\sigma_b^2, \sigma_d^2) > 0.$$

In addition, we also have from (9.8) that

$$\begin{aligned} c_n^2 \det(\Delta_n) &= (\alpha c_n - P(X_n))(\alpha c_n - R(X_n)) - Q^2(X_n), \\ &= \alpha c_n (\alpha c_n - P(X_n) - R(X_n)) + P(X_n)R(X_n) - Q^2(X_n), \\ &\geq P(X_k)R(X_k) + \alpha c_n^2 \max(|\rho_{ab}|, |\rho_{cd}|) - Q^2(X_n), \\ &\geq P(X_k)R(X_k) + \max(|\rho_{ab}|, |\rho_{cd}|)^2 c_n^2 - Q^2(X_n) > 0. \end{aligned}$$

Consequently, Δ_n is definite positive which immediately implies that $L_n \leq \alpha \Psi_n \Psi_n^t$. Moreover, we can use Lemma B.1 of [3] to say that

$$\Sigma_{n-1}^{-1} \Psi_n \Psi_n^t \Sigma_n^{-1} \leq \Sigma_{n-1}^{-1} - \Sigma_n^{-1}.$$

Hence

$$\begin{aligned} \mathbb{E}[(\Delta \mathcal{B}_{n+1})^2 | \mathcal{F}_n] &= 4M_n^t \Sigma_n^{-1} L_n \Sigma_n^{-1} M_n \quad \text{a.s.} \\ &\leq 4\alpha M_n^t \Sigma_n^{-1} \Psi_n \Psi_n^t \Sigma_n^{-1} M_n \quad \text{a.s.} \\ &\leq 4\alpha M_n^t (\Sigma_{n-1}^{-1} - \Sigma_n^{-1}) M_n \quad \text{a.s.} \end{aligned}$$

leading to $\langle \mathcal{B} \rangle_n \leq 4\alpha \mathcal{A}_n$. Therefore it follows from the strong law of large numbers for martingales that $\mathcal{B}_n = o(\mathcal{A}_n)$. Hence, we deduce from decomposition (9.1) that

$$\mathcal{V}_{n+1} + \mathcal{A}_n = o(\mathcal{A}_n) + \mathcal{O}(n) \quad \text{a.s.}$$

leading to $\mathcal{V}_{n+1} = \mathcal{O}(n)$ and $\mathcal{A}_n = \mathcal{O}(n)$ a.s. which implies that $\mathcal{B}_n = o(n)$ a.s. Finally we clearly obtain convergence (9.4) from the main decomposition (9.1) together with (9.2) and 9.3, which completes the proof of Lemma 9.1. \square

Lemma 9.2. *Assume that (H.1) to (H.4) are satisfied. For all $\delta > 1/2$, we have*

$$\|M_n\|^2 = o(|\mathbb{T}_n| n^\delta) \quad \text{a.s.}$$

Proof. Let us recall that

$$M_n = \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{c_k} \begin{pmatrix} X_k V_{2k} \\ V_{2k} \\ X_k V_{2k+1} \\ V_{2k+1} \end{pmatrix}.$$

Denote

$$P_n = \sum_{k \in \mathbb{T}_{n-1}} \frac{X_k V_{2k}}{c_k} \quad \text{and} \quad Q_n = \sum_{i \in \mathbb{T}_{n-1}} \frac{V_{2k}}{c_k}.$$

On the one hand, P_n can be rewritten as

$$P_n = \sum_{k=1}^n \sqrt{|\mathbb{G}_{k-1}|} f_k \quad \text{where} \quad f_n = \frac{1}{\sqrt{|\mathbb{G}_{n-1}|}} \sum_{k \in \mathbb{G}_{n-1}} \frac{X_k V_{2k}}{c_k}.$$

We already saw in Section 3 that for all $k \in \mathbb{G}_n$,

$$\mathbb{E}[V_{2k}|\mathcal{F}_n] = 0 \quad \text{and} \quad \mathbb{E}[V_{2k}^2|\mathcal{F}_n] = \sigma_a^2 X_k^2 + \sigma_c^2 = P(X_k) \quad \text{a.s.}$$

In addition, for all $k \in \mathbb{G}_n$,

$$\mathbb{E}[V_{2k}^4|\mathcal{F}_n] = \mu_a^4 X_k^4 + 6\sigma_a^2 \sigma_c^2 X_k^2 + \mu_c^4 \quad \text{a.s.}$$

which implies that

$$(9.9) \quad \mathbb{E}[V_{2k}^4|\mathcal{F}_n] \leq \mu_{ac}^4 c_k^2 \quad \text{a.s.}$$

where $\mu_{ac}^4 = \max(\mu_a^4, 3\sigma_a^2 \sigma_c^2, \mu_c^4)$. Consequently, $\mathbb{E}[f_{n+1}|\mathcal{F}_n] = 0$ a.s. and we deduce from (9.9) together with the Cauchy-Schwarz inequality that

$$\begin{aligned} \mathbb{E}[f_{n+1}^4|\mathcal{F}_n] &= \frac{1}{|\mathbb{G}_n|} \mathbb{E} \left[\left(\sum_{k \in \mathbb{G}_n} \frac{X_k V_{2k}}{c_k} \right)^4 \middle| \mathcal{F}_n \right], \\ &= \frac{1}{|\mathbb{G}_n|^2} \sum_{k \in \mathbb{G}_n} \left(\frac{X_k}{\sqrt{c_k}} \right)^4 \frac{\mathbb{E}[V_{2k}^4|\mathcal{F}_n]}{c_k^2} \\ &\quad + \frac{3}{|\mathbb{G}_n|^2} \sum_{k \in \mathbb{G}_n} \sum_{\substack{l \in \mathbb{G}_n \\ l \neq k}} \left(\frac{X_k}{\sqrt{c_k}} \right)^2 \left(\frac{X_l}{\sqrt{c_l}} \right)^2 \frac{\mathbb{E}[V_{2k}^2|\mathcal{F}_n]}{c_k} \frac{\mathbb{E}[V_{2l}^2|\mathcal{F}_n]}{c_l}, \\ &\leq \frac{1}{|\mathbb{G}_n|^2} \sum_{k \in \mathbb{G}_n} \mu_{ac}^4 + \frac{3}{|\mathbb{G}_n|^2} \sum_{k \in \mathbb{G}_n} \sum_{\substack{l \in \mathbb{G}_n \\ l \neq k}} \max(\sigma_a^2, \sigma_c^2)^2, \\ (9.10) \quad &\leq \mu_{ac}^4 + 3 \max(\sigma_a^2, \sigma_c^2)^2 \quad \text{a.s.} \end{aligned}$$

Therefore, we infer from (9.10) that

$$\sup_{n \geq 0} \mathbb{E}[f_{n+1}^4|\mathcal{F}_n] < \infty \quad \text{a.s.}$$

Hence, we obtain from Wei's Lemma given in [21] page 1672 that for all $\delta > 1/2$,

$$P_n^2 = o(|\mathbb{T}_{n-1}| n^\delta) \quad \text{a.s.}$$

On the other hand, Q_n can be rewritten as

$$Q_n = \sum_{k=1}^n \sqrt{|\mathbb{G}_{k-1}|} g_k \quad \text{where} \quad g_n = \frac{1}{\sqrt{|\mathbb{G}_{n-1}|}} \sum_{k \in \mathbb{G}_{n-1}} \frac{V_{2k}}{c_k}.$$

Via the same calculation as before, $\mathbb{E}[g_{n+1}|\mathcal{F}_n] = 0$ a.s. and, as $c_n \geq 1$,

$$\mathbb{E}[g_{n+1}^4|\mathcal{F}_n] \leq \mu_{bd}^4 + 3 \max(\sigma_b^2, \sigma_d^2)^2 \quad \text{a.s.}$$

Hence, we deduce once again from Wei's Lemma that for all $\delta > 1/2$,

$$Q_n^2 = o(|\mathbb{T}_{n-1}|n^\delta) \quad \text{a.s.}$$

In the same way, we obtain the same result for the two last components of M_n , which completes the proof of Lemma 9.2. \square

Proof of Theorem 5.4. We recall from (4.1) that $\hat{\theta}_n - \theta = \Sigma_{n-1}^{-1} M_n$ which implies

$$\|\hat{\theta}_n - \theta\|^2 \leq \frac{\mathcal{V}_n}{\lambda_{\min}(\Sigma_{n-1})}$$

where $\mathcal{V}_n = M_n^t \Sigma_{n-1}^{-1} M_n$. On the one hand, it follows from (9.4) that $\mathcal{V}_n = \mathcal{O}(n)$ a.s. On the other hand, we deduce from (9.7) that

$$\lim_{n \rightarrow \infty} \frac{\lambda_{\min}(\Sigma_n)}{|\mathbb{T}_n|} = \lambda_{\min}(C) > 0 \quad \text{a.s.}$$

Consequently, we find that

$$\|\hat{\theta}_n - \theta\|^2 = \mathcal{O}\left(\frac{n}{|\mathbb{T}_{n-1}|}\right) \quad \text{a.s.}$$

We are now in position to prove the quadratic strong law (5.4). First of all a direct application of Lemma 9.2 ensures that $\mathcal{V}_n = o(n^\delta)$ a.s. for all $\delta > 1/2$. Hence, we obtain from (9.4) that

$$(9.11) \quad \lim_{n \rightarrow \infty} \frac{\mathcal{A}_n}{n} = \frac{1}{2} \text{tr}((I_2 \otimes C)^{-1/2} L (I_2 \otimes C)^{-1/2}) \quad \text{a.s.}$$

Let us rewrite \mathcal{A}_n as

$$\mathcal{A}_n = \sum_{k=1}^n M_k^t (\Sigma_{k-1}^{-1} - \Sigma_k^{-1}) M_k = \sum_{k=1}^n M_k^t \Sigma_{k-1}^{-1/2} A_k \Sigma_{k-1}^{-1/2} M_k$$

where $A_k = I_4 - \Sigma_{k-1}^{1/2} \Sigma_k^{-1} \Sigma_{k-1}^{1/2}$. We already saw from (9.7) that

$$\lim_{n \rightarrow \infty} \frac{\Sigma_n}{|\mathbb{T}_n|} = I_2 \otimes C \quad \text{a.s.}$$

which ensures that

$$\lim_{n \rightarrow \infty} A_n = \frac{1}{2} I_4 \quad \text{a.s.}$$

In addition, we deduce from (9.4) that $\mathcal{A}_n = \mathcal{O}(n)$ a.s. which implies that

$$(9.12) \quad \frac{\mathcal{A}_n}{n} = \left(\frac{1}{2n} \sum_{k=1}^n M_k^t \Sigma_{k-1}^{-1} M_k \right) + o(1) \quad \text{a.s.}$$

Moreover we have

$$\begin{aligned}
\frac{1}{n} \sum_{k=1}^n M_k^t \Sigma_{k-1}^{-1} M_k &= \frac{1}{n} \sum_{k=1}^n (\hat{\theta}_k - \theta)^t \Sigma_{k-1} (\hat{\theta}_k - \theta), \\
&= \frac{1}{n} \sum_{k=1}^n |\mathbb{T}_{k-1}| (\hat{\theta}_k - \theta)^t \frac{\Sigma_{k-1}}{|\mathbb{T}_{k-1}|} (\hat{\theta}_k - \theta), \\
(9.13) \quad &= \frac{1}{n} \sum_{k=1}^n |\mathbb{T}_{k-1}| (\hat{\theta}_k - \theta)^t (I_2 \otimes C) (\hat{\theta}_k - \theta) + o(1) \quad \text{a.s.}
\end{aligned}$$

Therefore, (9.11) together with (9.12) and (9.13) lead to (5.4).

10. PROOF OF THEOREM 5.5

First of all, we shall only prove (5.6) since the proof of (5.7) follows exactly the same lines. We clearly have from (3.7) that

$$\begin{aligned}
Q_{n-1}(\hat{\eta}_n - \eta_n) &= \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} (\hat{V}_{2k}^2 - V_{2k}^2) \psi_k, \\
&= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k} (\hat{V}_{2k}^2 - V_{2k}^2) \psi_k, \\
(10.1) \quad &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k} \left((\hat{V}_{2k} - V_{2k})^2 + 2(\hat{V}_{2k} - V_{2k}) V_{2k} \right) \psi_k.
\end{aligned}$$

In addition, we already saw in Section 3 that for all $l \geq 0$ and $k \in \mathbb{G}_l$,

$$\hat{V}_{2k} - V_{2k} = - \begin{pmatrix} \hat{a}_l - a \\ \hat{c}_l - c \end{pmatrix}^t \Phi_k.$$

Consequently,

$$(\hat{V}_{2k} - V_{2k})^2 \leq \|\Phi_k\|^2 ((\hat{a}_l - a)^2 + (\hat{c}_l - c)^2) = c_k ((\hat{a}_l - a)^2 + (\hat{c}_l - c)^2).$$

Hence, as $\|\psi_k\|^2 = X_k^4 + 1 \leq c_k^2$,

$$\begin{aligned}
\left\| \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{(\hat{V}_{2k} - V_{2k})^2}{d_k} \psi_k \right\| &\leq \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{c_k \|\psi_k\|}{d_k} ((\hat{a}_l - a)^2 + (\hat{c}_l - c)^2), \\
&\leq \sum_{l=0}^{n-1} |\mathbb{G}_l| ((\hat{a}_l - a)^2 + (\hat{c}_l - c)^2).
\end{aligned}$$

However, as Λ is positive definite, we obtain from (5.4) that

$$\sum_{l=0}^{n-1} |\mathbb{G}_l| ((\hat{a}_l - a)^2 + (\hat{c}_l - c)^2) = \mathcal{O}(n) \quad \text{a.s.}$$

which implies that

$$(10.2) \quad \left\| \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{(\widehat{V}_{2k} - V_{2k})^2}{d_k} \psi_k \right\| = \mathcal{O}(n) \quad \text{a.s.}$$

Furthermore, denote

$$P_n = \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{(\widehat{V}_{2k} - V_{2k})V_{2k}}{d_k} \psi_k.$$

We clearly have

$$\begin{aligned} \Delta P_{n+1} &= P_{n+1} - P_n = \sum_{k \in \mathbb{G}_n} \frac{(\widehat{V}_{2k} - V_{2k})V_{2k}}{d_k} \psi_k, \\ &= - \sum_{k \in \mathbb{G}_n} \frac{V_{2k}}{d_k} \psi_k \Phi_k^t \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix}. \end{aligned}$$

In addition, for all $k \in \mathbb{G}_n$, $\mathbb{E}[V_{2k}|\mathcal{F}_n] = 0$ a.s. and $\mathbb{E}[V_{2k}^2|\mathcal{F}_n] = \sigma_a^2 X_k + \sigma_c^2 \leq \alpha c_k$ a.s. where $\alpha = \max(\sigma_a^2, \sigma_c^2)$. Consequently, $\mathbb{E}[\Delta P_{n+1}|\mathcal{F}_n] = 0$ a.s. and

$$\begin{aligned} \mathbb{E}[\Delta P_{n+1} \Delta P_{n+1}^t | \mathcal{F}_n] &= \sum_{k \in \mathbb{G}_n} \frac{1}{d_k^2} \mathbb{E}[V_{2k}^2 | \mathcal{F}_n] \psi_k \Phi_k^t \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix} \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix}^t \Phi_k \psi_k^t \quad \text{a.s.} \\ &= \sum_{k \in \mathbb{G}_n} \frac{P(X_k)}{d_k^2} \psi_k \Phi_k^t \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix} \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix}^t \Phi_k \psi_k^t \quad \text{a.s.} \end{aligned}$$

Therefore, (P_n) is a square integrable vector martingale with increasing process $\langle P \rangle_n$ given by

$$\begin{aligned} \langle P \rangle_n &= \sum_{l=1}^{n-1} \mathbb{E}[\Delta P_{l+1} \Delta P_{l+1}^t | \mathcal{F}_l] \quad \text{a.s.} \\ &= \sum_{l=1}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{P(X_k)}{d_k^2} \psi_k \Phi_k^t \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix} \begin{pmatrix} \widehat{a}_l - a \\ \widehat{c}_l - c \end{pmatrix}^t \Phi_k \psi_k^t \quad \text{a.s.} \end{aligned}$$

It immediately follows from the previous calculation that

$$\begin{aligned} \|\langle P \rangle_n\| &\leq \alpha \sum_{l=0}^{n-1} ((\widehat{a}_l - a)^2 + (\widehat{c}_l - c)^2) \sum_{k \in \mathbb{G}_l} \frac{c_k \|\psi_k\|^2 \|\Phi_k\|^2}{d_k^2} \quad \text{a.s.} \\ &\leq \alpha \sum_{l=0}^{n-1} |\mathbb{G}_l| ((\widehat{a}_l - a)^2 + (\widehat{c}_l - c)^2) \quad \text{a.s.} \end{aligned}$$

leading to

$$\|\langle P \rangle_n\| = \mathcal{O}(n) \quad \text{a.s.}$$

Then, we deduce from the strong law of large numbers for martingale given e.g. in Theorem 1.3.15 of [8] that

$$(10.3) \quad P_n = o(n) \quad \text{a.s.}$$

Hence, we find from (10.1), (10.2) and (10.3) that

$$\|Q_{n-1}(\widehat{\eta}_n - \eta_n)\| = \mathcal{O}(n) \quad \text{a.s.}$$

Moreover, we infer once again from Lemma 5.2 that

$$(10.4) \quad \lim_{n \rightarrow \infty} \frac{1}{|\mathbb{T}_n|} Q_n = D = \mathbb{E} \left[\frac{1}{(1 + T^2)^2} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right] \quad \text{a.s.}$$

which ensures, since D is positive definite, that

$$\|\widehat{\eta}_n - \eta_n\| = \mathcal{O} \left(\frac{n}{|\mathbb{T}_{n-1}|} \right) \quad \text{a.s.}$$

It remains to establish (5.8). Denote

$$\widehat{W}_n = \begin{pmatrix} \widehat{V}_{2n} \\ \widehat{V}_{2n+1} \end{pmatrix} \quad \text{and} \quad R_n = \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} \left(\widehat{W}_k - W_k \right)^t J W_k \psi_k$$

where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, we have from (3.9) that

$$Q_{n-1}(\widehat{\nu}_n - \nu_n) = \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} \left(\widehat{V}_{2k} - V_{2k} \right) \left(\widehat{V}_{2k+1} - V_{2k+1} \right) \psi_k + R_n.$$

It is not hard to see that (R_n) is a square integrable real martingale with increasing process given by

$$\begin{aligned} \langle R \rangle_n &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \mathbb{E} \left[\frac{1}{d_k^2} \left(\widehat{W}_k - W_k \right)^t J W_k W_k^t J \left(\widehat{W}_k - W_k \right) \psi_k \psi_k^t \middle| \mathcal{F}_n \right] \quad \text{a.s.} \\ &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k^2} \left(\widehat{W}_k - W_k \right)^t J \mathbb{E} \left[W_k W_k^t \middle| \mathcal{F}_n \right] J \left(\widehat{W}_k - W_k \right) \psi_k \psi_k^t \quad \text{a.s.} \\ &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k^2} \left(\widehat{W}_k - W_k \right)^t J \begin{pmatrix} P(X_k) & Q(X_k) \\ Q(X_k) & R(X_k) \end{pmatrix} J \left(\widehat{W}_k - W_k \right) \psi_k \psi_k^t \quad \text{a.s.} \\ &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k^2} \left(\widehat{W}_k - W_k \right)^t \begin{pmatrix} R(X_k) & Q(X_k) \\ Q(X_k) & P(X_k) \end{pmatrix} \left(\widehat{W}_k - W_k \right) \psi_k \psi_k^t \quad \text{a.s.} \end{aligned}$$

Consequently, Lemma 5.2 together with (5.4) allows us to say that $\| \langle R \rangle_n \| = \mathcal{O}(n)$ a.s. which ensures that $R_n = o(n)$ a.s. Moreover,

$$\begin{aligned}
& \left\| \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} \left(\widehat{V}_{2k} - V_{2k} \right) \left(\widehat{V}_{2k+1} - V_{2k+1} \right) \psi_k \right\| \\
& \leq \frac{1}{2} \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} \left(\left(\widehat{V}_{2k} - V_{2k} \right)^2 + \left(\widehat{V}_{2k+1} - V_{2k+1} \right)^2 \right) \|\psi_k\|, \\
& \leq \frac{1}{2} \sum_{l=0}^{n-1} \|\widehat{\theta}_l - \theta\|^2 \sum_{k \in \mathbb{G}_l} \frac{\|\Phi_k\|^2 \|\psi_k\|}{d_k}, \\
& \leq \frac{1}{2} \sum_{l=0}^{n-1} |\mathbb{G}_l| \|\widehat{\theta}_l - \theta\|^2,
\end{aligned}$$

which implies via (5.4) that

$$\left\| \sum_{k \in \mathbb{T}_{n-1}} \left(\widehat{V}_{2k} - V_{2k} \right) \left(\widehat{V}_{2k+1} - V_{2k+1} \right) \psi_k \right\| = \mathcal{O}(n) \quad \text{a.s.}$$

Therefore, we obtain that

$$|\mathbb{T}_{n-1}| \|\widehat{\nu}_n - \nu_n\| = \mathcal{O}(n) \quad \text{a.s.}$$

which leads to (5.8). Finally, it only remains to prove the a.s. convergence of η_n , ζ_n and ν_n to η , ζ and ν which will immediately lead to the a.s. convergence of $\widehat{\eta}_n$, $\widehat{\zeta}_n$ and $\widehat{\nu}_n$ through (5.6), (5.7) and (5.8), respectively. On the one hand,

$$(10.5) \quad Q_{n-1}(\eta_n - \eta) = N_n = \sum_{k \in \mathbb{T}_n} \frac{1}{d_k} v_{2k} \psi_k$$

where we recall that $v_{2n} = V_{2n}^2 - \eta^t \psi_n$. It is clear that (N_n) is a square integrable vector martingale with increasing process $\langle N \rangle_n$ given by

$$\begin{aligned}
\langle N \rangle_n &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k^2} \mathbb{E}[v_{2k}^2 | \mathcal{F}_l] \psi_k \psi_k^t \quad \text{a.s.} \\
&\leq \gamma \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k} \psi_k \psi_k^t \quad \text{a.s.}
\end{aligned}$$

where $\gamma = \max(\mu_a^4 - \sigma_a^4, 2\sigma_a^2 \sigma_c^2, \mu_c^4 - \sigma_c^4)$. Hence,

$$\| \langle N \rangle_n \| \leq \gamma \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k} \|\psi_k\|^2 \leq \gamma |\mathbb{T}_n| \quad \text{a.s.}$$

which immediately leads to

$$\| \langle N \rangle_n \| = \mathcal{O}(|\mathbb{T}_{n-1}|) \quad \text{a.s.}$$

Consequently,

$$\|N_n\|^2 = \mathcal{O}(n|\mathbb{T}_{n-1}|) \quad \text{a.s.}$$

which leads via (10.4) and (10.5) to the a.s. convergence of η_n to η and to the rate of convergence of Remark 5.6. The proof of the a.s. convergence of ζ_n to ζ follows exactly the same lines. On the other hand

$$(10.6) \quad Q_{n-1}(\nu_n - \nu) = H_n = \sum_{k \in \mathbb{T}_{n-1}} \frac{1}{d_k} w_{2k} \psi_k$$

where we recall that $w_{2k} = V_{2k}V_{2k+1} - \mathbb{E}[V_{2k}V_{2k+1}|\mathcal{F}_n]$. It is obvious to see that (H_n) is a square integrable real martingale with increasing process

$$\begin{aligned} \langle H \rangle_n &= \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k^2} \mathbb{E}[w_{2k}^2 | \mathcal{F}_l] \psi_k \psi_k^t, \\ &\leq \alpha \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k} \psi_k \psi_k^t, \end{aligned}$$

where $\alpha = \max(\nu_{ab}^2, \nu_{cd}^2, (\sigma_a^2 + \sigma_c^2)(\sigma_b^2 + \sigma_d^2))$. This implies that

$$\|\langle H \rangle_n\| \leq \alpha \sum_{l=0}^{n-1} \sum_{k \in \mathbb{G}_l} \frac{1}{d_k} \|\psi_k\|^2 \leq \alpha |\mathbb{T}_{n-1}|$$

which allows us to say that

$$\|H_n\|^2 = \mathcal{O}(n|\mathbb{T}_{n-1}|) \quad \text{and} \quad \|\hat{\nu}_n - \nu\|^2 = \mathcal{O}\left(\frac{n}{|\mathbb{T}_{n-1}|}\right) \quad \text{a.s.}$$

Finally, we deduce from (10.6) that ν_n goes a.s. to ν and that the rate of convergence of Remark 5.6 is verified, which completes the proof of Theorem 5.5.

11. PROOF OF THEOREM 5.7

In order to establish the asymptotic normality of our estimators, we will extensively make use of the central limit theorem for triangular arrays of vector martingales given e.g. by Theorem 2.1.9 of [8]. First of all, instead of using the generation-wise filtration (\mathcal{F}_n) , we will use the sister pair-wise filtration (\mathcal{G}_n) given by

$$\mathcal{G}_n = \sigma(X_1, (X_{2k}, X_{2k+1}), 1 \leq k \leq n).$$

Proof of Theorem 5.7, first part. We focus our attention to the proof of the asymptotic normality (5.9). Let $M^{(n)} = (M_k^{(n)})$ be the square integrable vector martingale defined as

$$(11.1) \quad M_k^{(n)} = \frac{1}{\sqrt{|\mathbb{T}_n|}} \sum_{i=1}^k D_i$$

where

$$D_i = \frac{1}{c_i} \begin{pmatrix} X_i V_{2i} \\ V_{2i} \\ X_i V_{2i+1} \\ V_{2i+1} \end{pmatrix}.$$

We clearly have

$$(11.2) \quad M_{t_n}^{(n)} = \frac{1}{\sqrt{|\mathbb{T}_n|}} \sum_{i=1}^{t_n} D_i = \frac{1}{\sqrt{|\mathbb{T}_n|}} M_{n+1}$$

where $t_n = |\mathbb{T}_n|$. Moreover, the increasing process associated to $(M_k^{(n)})$ is given by

$$\begin{aligned} \langle M^{(n)} \rangle_k &= \frac{1}{|\mathbb{T}_n|} \sum_{i=1}^k \mathbb{E} [D_i D_i^t | \mathcal{G}_{i-1}], \\ &= \frac{1}{|\mathbb{T}_n|} \sum_{i=1}^k \frac{1}{c_i^2} \begin{pmatrix} P(X_i) & Q(X_i) \\ Q(X_i) & R(X_i) \end{pmatrix} \otimes \begin{pmatrix} X_i^2 & X_i \\ X_i & 1 \end{pmatrix} \quad \text{a.s.} \end{aligned}$$

Consequently, it follows from convergence (5.3) that

$$\lim_{n \rightarrow \infty} \langle M^{(n)} \rangle_{t_n} = L \quad \text{a.s.}$$

It is now necessary to verify Lindeberg's condition by use of Lyapunov's condition. Denote

$$\phi_n = \sum_{k=1}^{t_n} \mathbb{E} \left[\|M_k^{(n)} - M_{k-1}^{(n)}\|^4 \middle| \mathcal{G}_{k-1} \right].$$

We obtain from (11.1) that

$$\begin{aligned} \phi_n &= \frac{1}{|\mathbb{T}_n|^2} \sum_{k=1}^{t_n} \mathbb{E} \left[\frac{(1 + X_k^2)^2}{c_k^4} (V_{2k}^2 + V_{2k+1}^2)^2 \middle| \mathcal{G}_{k-1} \right], \\ &\leq \frac{2}{|\mathbb{T}_n|^2} \sum_{k=1}^{t_n} \frac{1}{c_k^2} (\mathbb{E}[V_{2k}^4 | \mathcal{G}_{k-1}] + \mathbb{E}[V_{2k+1}^4 | \mathcal{G}_{k-1}]). \end{aligned}$$

In addition, we already saw in Section 9 that

$$\mathbb{E}[V_{2n}^4 | \mathcal{G}_{n-1}] \leq \mu_{ac}^4 c_n^2, \quad \mathbb{E}[V_{2n+1}^4 | \mathcal{G}_{n-1}] \leq \mu_{bd}^4 c_n^2 \quad \text{a.s.}$$

where $\mu_{ac}^4 = \max(\mu_a^4, 3\sigma_a^2 \sigma_c^2, \mu_c^4)$ and $\mu_{bd}^4 = \max(\mu_b^4, 3\sigma_b^2 \sigma_d^2, \mu_d^4)$. Hence,

$$\phi_n \leq \frac{2(\mu_{ac}^4 + \mu_{bd}^4)}{|\mathbb{T}_n|} \quad \text{a.s.}$$

which immediately implies that

$$\lim_{n \rightarrow \infty} \phi_n = 0 \quad \text{a.s.}$$

Therefore, Lyapunov's condition is satisfied and Theorem 2.1.9 of [8] allows us to say via (11.2) that

$$\frac{1}{\sqrt{|\mathbb{T}_{n-1}|}} M_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, L).$$

Finally, we infer from (4.1) together with (9.7) and Slutsky's lemma that

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Lambda^{-1} L \Lambda^{-1}). \quad \square$$

Proof of Theorem 5.7, second part. We shall now establish the asymptotic normality given by (5.10). Denote by $N^{(n)} = (N_k^{(n)})$ the square integrable vector martingale defined as

$$N_k^{(n)} = \frac{1}{\sqrt{|\mathbb{T}_n|}} \sum_{i=1}^k \frac{v_{2i}}{d_i} \psi_i.$$

We immediately see from (10.5) that

$$(11.3) \quad N_{t_n}^{(n)} = \frac{1}{\sqrt{|\mathbb{T}_n|}} Q_n(\eta_{n+1} - \eta) = \frac{1}{\sqrt{|\mathbb{T}_n|}} N_{n+1}.$$

In addition, the increasing process associated to $(N_k^{(n)})$ is given by

$$\begin{aligned} \langle N^{(n)} \rangle_k &= \frac{1}{|\mathbb{T}_n|} \sum_{i=1}^k \mathbb{E} \left[\frac{v_{2i}^2}{d_i^2} \psi_i \psi_i^t \middle| \mathcal{G}_{i-1} \right], \\ &= \frac{1}{|\mathbb{T}_n|} \sum_{i=1}^k \frac{(\mu_a^4 - \sigma_a^4) X_i^4 + 4\sigma_a^2 \sigma_c^2 X_i^2 + (\mu_c^4 - \sigma_c^4)}{d_i^2} \psi_i \psi_i^t \quad \text{a.s.} \end{aligned}$$

Consequently, we obtain from Lemma 5.2 that

$$\lim_{n \rightarrow \infty} \langle N^{(n)} \rangle_{t_n} = \mathbb{E} \left[\frac{(\mu_a^4 - \sigma_a^4) T^4 + 4\sigma_a^2 \sigma_c^2 T^2 + (\mu_c^4 - \sigma_c^4)}{(1 + T^2)^4} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right] = M_{ac} \quad \text{a.s.}$$

In order to verify Lyapunov's condition, let

$$\phi_n = \sum_{k=1}^{t_n} \mathbb{E} \left[\|N_k^{(n)} - N_{k-1}^{(n)}\|^3 \middle| \mathcal{G}_{k-1} \right].$$

We clearly have

$$\|N_k^{(n)} - N_{k-1}^{(n)}\|^2 = \frac{1}{|\mathbb{T}_n|} \frac{v_{2k}^2}{d_k^2} \|\psi_k\|^2 \leq \frac{1}{|\mathbb{T}_n|} \frac{v_{2k}^2}{d_k},$$

which implies that

$$\|N_k^{(n)} - N_{k-1}^{(n)}\|^3 \leq \frac{1}{|\mathbb{T}_n|^{3/2}} \frac{|v_{2k}|^3}{d_k^{3/2}} = \frac{1}{|\mathbb{T}_n|^{3/2}} \frac{|v_{2k}|^3}{c_k^3}.$$

However,

$$\begin{aligned} |v_{2k}|^3 &= |V_{2k}^2 - \sigma_a^2 X_k^2 - \sigma_c^2|^3 \leq (V_{2k}^2 + \sigma_a^2 X_k^2 + \sigma_c^2)^3, \\ (11.4) \quad &\leq V_{2k}^6 + 3V_{2k}^4(\sigma_a^2 X_k^2 + \sigma_c^2) + 3V_{2k}^2(\sigma_a^2 X_k^2 + \sigma_c^2)^2 + (\sigma_a^2 X_k^2 + \sigma_c^2)^3. \end{aligned}$$

We already saw that $\mathbb{E}[V_{2k}^2|\mathcal{G}_{k-1}] = \sigma_a^2 X_k + \sigma_c^2$ a.s. and that $\mathbb{E}[V_{2k}^4|\mathcal{G}_{k-1}] \leq \mu_{ac} c_k^2$ a.s. Moreover, it can easily be seen that

$$\mathbb{E}[V_{2k}^6|\mathcal{G}_{k-1}] \leq \tau_{ac}^6 c_k^3 \quad \text{a.s.}$$

where $\tau_{ac}^6 = \max\left(\tau_a^6 + 10\sqrt{\tau_a^6 \tau_c^6}, 5\mu_a^4 \sigma_c^2, 5\mu_c^4 \sigma_a^2, \tau_c^6 + 10\sqrt{\tau_a^6 \tau_c^6}\right)$. Consequently, we deduce from (11.4) that it exists some constant $\gamma > 0$ such that

$$\mathbb{E}[|v_{2k}|^3|\mathcal{G}_{k-1}] \leq \gamma c_k^3 \quad \text{a.s.}$$

Then, we can conclude that

$$\phi_n \leq \frac{\gamma}{\sqrt{|\mathbb{T}_n|}} \quad \text{a.s.}$$

which immediately leads to

$$\lim_{n \rightarrow \infty} \phi_n = 0 \quad \text{a.s.}$$

Therefore, Lyapunov's condition is satisfied and we find from Theorem 2.1.9 of [8] and (11.3) that

$$(11.5) \quad \frac{1}{\sqrt{|\mathbb{T}_{n-1}|}} N_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, M_{ac}).$$

Hence, we obtain from (10.4), (11.5) and Slutsky's lemma that

$$\sqrt{|\mathbb{T}_{n-1}|}(\eta_n - \eta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1} M_{ac} D^{-1}).$$

Finally, (5.6) ensures that

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\eta}_n - \eta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1} M_{ac} D^{-1}).$$

The proof of (5.11) follows exactly the same lines. \square

Proof of Theorem 5.7, third part. It remains to establish the asymptotic normality given by (5.12). Denote by $H^{(n)} = (H_k^{(n)})$ the square integrable martingale defined as

$$H_k^{(n)} = \frac{1}{\sqrt{|\mathbb{T}_n|}} \sum_{i=1}^k \frac{w_{2i}}{d_i} \psi_i.$$

We clearly have

$$H_{t_n}^{(n)} = \frac{1}{\sqrt{|\mathbb{T}_n|}} \sum_{i=1}^{t_n} \frac{w_{2i}}{d_i} \psi_i = \frac{1}{\sqrt{|\mathbb{T}_n|}} H_{n+1}.$$

Moreover, the increasing process of $(H_k^{(n)})$ is given by

$$\langle H^{(n)} \rangle_k = \frac{1}{|\mathbb{T}_n|} \sum_{i=1}^k \frac{\mathbb{E}[w_{2i}^2|\mathcal{G}_{i-1}] \psi_i \psi_i^t}{d_i^2}.$$

In addition, we already saw in Section 3 that

$$\mathbb{E}[w_{2k}^2|\mathcal{F}_n] = (\nu_{ab}^2 - \rho_{ab}^2) X_k^4 + (\sigma_a^2 \sigma_d^2 + \sigma_b^2 \sigma_c^2 + 2\rho_{ab} \rho_{cd}) X_k^2 + (\nu_{cd}^2 - \rho_{cd}^2) \quad \text{a.s.}$$

Then, we deduce once again from Lemma 5.2 that

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle H^{(n)} \rangle_{t_n} &= \mathbb{E} \left[\frac{(\nu_{ab}^2 - \rho_{ab}^2)T^4 + (\sigma_a^2 \sigma_d^2 + \sigma_b^2 \sigma_c^2 + 2\rho_{ab}\rho_{cd})T^2 + (\nu_{cd}^2 - \rho_{cd}^2)}{(1 + T^2)^4} \begin{pmatrix} T^4 & T^2 \\ T^2 & 1 \end{pmatrix} \right] \\ &= H \quad \text{a.s.} \end{aligned}$$

In order to verify Lyapunov's condition, denote

$$\phi_n = \sum_{k=1}^{t_n} \mathbb{E} \left[\|H_k^{(n)} - H_{k-1}^{(n)}\|^3 \middle| \mathcal{G}_{k-1} \right].$$

As in the previous proof, we clearly have that

$$\|H_k^{(n)} - H_{k-1}^{(n)}\|^3 \leq \frac{1}{|\mathbb{T}_n|^{3/2}} \frac{|w_{2k}|^3}{c_k^3}.$$

We can observe that

$$\begin{aligned} |w_{2k}|^3 &= |V_{2k}V_{2k+1} - \rho_{ab}X_k^2 - \rho_{cd}|^3 \leq (|V_{2k}V_{2k+1}| + |\rho_{ab}|X_k^2 + |\rho_{cd}|)^3, \\ &\leq \frac{1}{2}(V_{2k}^6 + V_{2k+1}^6) + 3V_{2k}^2V_{2k+1}^2(|\rho_{ab}|X_k^2 + |\rho_{cd}|) \\ &\quad + \frac{3}{2}(V_{2k}^2 + V_{2k+1}^2)(|\rho_{ab}|X_k^2 + |\rho_{cd}|)^2 + (|\rho_{ab}|X_k^2 + |\rho_{cd}|)^3. \end{aligned}$$

Hence, it follows from the previous calculations that it exists some constant $\xi > 0$ such that

$$\mathbb{E}[|w_k|^3 | \mathcal{G}_{k-1}] \leq \xi c_k^3 \quad \text{a.s.}$$

which immediately leads to

$$\mathbb{E}[\|H_k^{(n)} - H_{k-1}^{(n)}\|^3 | \mathcal{G}_{k-1}] \leq \frac{\xi}{|\mathbb{T}_n|^{3/2}} \quad \text{a.s.}$$

which ensures that

$$\phi_n \leq \frac{\xi}{\sqrt{|\mathbb{T}_n|}} \quad \text{a.s.}$$

Then, we obviously have that

$$\lim_{n \rightarrow \infty} \phi_n = 0 \quad \text{a.s.}$$

and we can conclude that

$$\frac{1}{\sqrt{|\mathbb{T}_{n-1}|}} H_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, H).$$

In other words

$$\sqrt{|\mathbb{T}_{n-1}|}(\nu_n - \nu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1}HD^{-1}).$$

Finally, we find via (5.8) that

$$\sqrt{|\mathbb{T}_{n-1}|}(\hat{\nu}_n - \nu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1}HD^{-1})$$

which achieves the proof of Theorem 5.7. \square

Acknowledgement. I would like to thank Bernard Bercu for his helpful suggestions and for thorough readings of the paper.

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